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NONMONOTONIC CAVITY GROWTH IN FINITE, COMPRESSIBLE ELASTICITY

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Abstract—We obtain closed form solutions to the problems of cylindrical and spherical cavitation when a stretch is prescribed on the outer boundary for a compressible elastic material. The strain energy function is quite general and contains an arbitrary function of a linear combination of the principal stretch invariants. The cavitating solution is shown to be preferred to the corresponding homogeneous deformation. Three modes of cavitation are identified for the spherical problem whereas only one is possible in the cylindrical case. The most general strain energy function for which the cavitating solution is possible is briefly discussed. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

In applications, it has been commonly recognised that highly elastic materials such as elastomers create cavities as precursors to failure. For instance, Gent and Lindley (1958) reported that many spherical cavities appeared in short rubber cylinders when subjected to tensile loading. Their specimens were such that the deformation of the rubber in the lateral direction was highly constrained and great triaxial stress could be easily achieved in the interior. They proceeded to propose a criterion that a cavity would form when the local triaxial stress reaches a certain value which is characteristic of the material. Gent and Tompkins (1969) observed that similar cavities were created in over-gas-saturated rubbers when the external pressure was removed, thus reconfirming the criterion put forward by Gent and Lindley (1958). Oberth and Bruenner (1965) also observed that in an elastomeric material, cavities were formed near stiffer inclusions by the high triaxial stress. More recently, cavitation in rubber particles dispersed in polymeric matrices is drawing increasing attention because of its connection with failure mechanisms and also toughening effects (e.g., Beahan *et al.*, 1976; Lazzeri and Bucknall, 1993 and Bucknall *et al.* 1994).

In most of the situations cited above, highly elastic materials are subjected to substantially constrained loading, resulting in large triaxial stress. In these circumstances the phenomenon of cavitation may be understood in the sense that the strain energy stored by volumetric expansion would be, when a critical level is exceeded, converted into distortional energy accompanying high equi-biaxial deformations around an initiated cavity. This interpretation may also be appropriate to the analysis of the bifurcation of deformations at a critical level of loading, which problem has been of major interest in the mechanics of solids.

The notion of bifurcation was emphasised by Ball (1982) who formulated the problem of modelling cavity formation in the context of nonlinear elasticity. Ball (1982) analysed deformations of a solid sphere of homogeneous and isotropic hyperelastic material subjected to uniform radial tension at the outer surface. The sphere remains intact if the tension is small enough. For certain popular material models, however, there also exists a competing solution which contains a spherical cavity at the centre when the tension exceeds a certain limit: a cavitating bifurcation occurs. In his major contribution, Ball (1982) gives an explicit formula for the critical stress in the case of incompressible hyperelastic solids, where the assumed constraint of incompressibility facilitates the analysis to a great extent. In the special case of neo-Hookean solids, the critical stress found was in agreement with the experimental criterion obtained by Gent and Lindley (1958).

However, it is a formidable task to analyse the corresponding problems for compressible solids in a generalised manner. Following Ball's example, mathematical issues relating to existence, uniqueness and stability of cavitating solutions were studied by Stuart (1985), Sivaloganathan (1986a, b) and Podio-Guidugli *et al.* (1986). Regarding the connection between cavitation phenomena and material constitutive behaviour, there have been numerous attempts to consider cavitation for specific hyperelastic material models. The critical values for cavitation in the plane-strain axisymmetric as well as spherically symmetric deformations of the so-called Blatz–Ko material (Blatz and Ko, 1962) were obtained by Horgan and Abeyaratne (1986). Further analysis was undertaken by Ertan (1988), Biwa *et al.* (1994) and Biwa (1995) incorporating hyperelastic potentials which generalise the Blatz–Ko model. Horgan (1992) analysed cavitation for the so-called generalised Varga materials where closed-form solutions for the cavity radius were obtained for both spherical and cylindrical deformations. This work illustrates the phenomenon of cavitation for compressible materials in a particularly tractable setting.

The major part of this paper generalises the work of Horgan (1992) using a hyperelastic potential which includes the generalised Varga materials as a special case. What is noteworthy in our approach is that the simplicity and elegance of the work of Horgan is maintained but in a more general setting. Furthermore we identify three possible modes of cavitation: the first is where the cavity radius increases monotonically with applied stretch (as is the case for the generalised Varga material), the second mode is where the cavity radius does not increase monotonically and the third is where with increasing applied stretch at the outer surface, a cavity first appears, then closes and finally reopens again. The latter two modes have not been previously identified in the literature.

Most previous work on the problem of cavitation is characterised by the following approach: assume some specified hyperelastic model, presumed to be valid for arbitrary strain range, use the equations of equilibrium to obtain the corresponding radial deformation field and then solve the boundary value problem associated with cavitation. This is, initially, our approach also but we also obtain the most general strain energy function for which our cavitation solution is possible, using recent results of Murphy (1997) which partly provided the motivation for the present analysis. As a consequence of this, we will show that the hyperelastic potential of the generalised Varga material is the most general for which the solution of Horgan (1992) is valid. Therefore, we believe this paper also contributes to the discussion as to the type of hyperelastic material which allows the phenomenon of cavitation to occur.

We finally remark that an excellent review of the problem of cavitation in nonlinear elasticity was provided recently by Horgan and Polignone (1995), where a more extensive list of references is given.

2. PRELIMINARIES

The response of an elastic material is described completely by the form of its strain energy function

$$W = \hat{W}(\mathbf{F}) \tag{1}$$

where $\mathbf{F}$ is the deformation gradient tensor satisfying

$$\det \mathbf{F} > 0. \tag{2}$$

We note that $\mathbf{F}$ has the polar decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{3}$$

where the rotation tensor $\mathbf{R}$ is a proper orthogonal tensor and the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are positive-definite and symmetric.

Invariance under superposed rigid-body motions leads to

$$W = \bar{W}(\mathbf{U}). \tag{4}$$

The assumption of material isotropy further leads to

$$W = \tilde{W}(i_1, i_2, i_3), \tag{5}$$

where i_1, i_2 and i_3 are the principal invariants of the stretch tensors.

The stress response equations

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{T} = i_3^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}, \tag{6}$$

where $\mathbf{P}, \mathbf{T}$ are the Piola and Cauchy stress tensors respectively, then lead to a representation

$$\mathbf{T} = \frac{\partial W}{\partial i_3}\mathbf{I} + i_3^{-1}\left(\frac{\partial W}{\partial i_1} + i_1\frac{\partial W}{\partial i_2}\right)\mathbf{V} - i_3^{-1}\frac{\partial W}{\partial i_2}\mathbf{V}^2. \tag{7}$$

Imposing the conditions that the strain energy and the stress vanish in the reference configuration, we obtain

$$W(i_1, i_2, i_3)|_{i_1 = i_2 = 3; i_3 = 1} = 0, \quad \left.\frac{\partial W}{\partial i_1} + 2\frac{\partial W}{\partial i_2} + \frac{\partial W}{\partial i_3}\right|_{i_1 = i_2 = 3; i_3 = 1} = 0. \tag{8}$$

3. THE CHOICE OF STRAIN ENERGY FUNCTION

Assume initially that the strain energy function has the form

$$W = \Omega(\alpha i_1 + \beta i_2 + \gamma i_3) + c_1 i_1 + c_2 i_2 + c_3 i_3, \tag{9}$$

where $\Omega(\cdot)$ is an arbitrary function and α, β, γ, c_1, c_2, c_3 are arbitrary constants. We first note that (9) is a generalisation of the three materials considered by Carroll (1988): setting $\beta = \gamma = 0$ recovers the harmonic material, setting $\alpha = \gamma = 0$ recovers the material of type II and setting $\alpha = \beta = 0$ recovers the generalised Varga material. In view of the work of Carroll we will call the material with the above strain energy function, the generalised Carroll (or GC) material. The generalised Varga material is the material studied by Horgan (1992) in the context of cavitation in unconstrained elasticity and our work will be a generalisation of the results obtained by him.

The problem of finding restrictions on the strain energy function such as to ensure physically realistic behaviour is an open one in finite elasticity. Two obvious necessary restrictions are the reference configuration restrictions given in (8). Imposing these on (9) we obtain

$$\Omega_0 + 3c_1 + 3c_2 + c_3 = 0, \quad (\alpha + 2\beta + \gamma)\Omega_0' + c_1 + 2c_2 + c_3 = 0, \tag{10}$$

where

$$\Omega_0 \equiv \Omega(3\alpha+3\beta+\gamma), \quad \Omega_0' \equiv \left.\frac{\mathrm{d}\Omega}{\mathrm{d}(\alpha i_1+\beta i_2+\gamma i_3)}\right|_{i_1=i_2=3;i_3=1}. \tag{11}$$

We also note that since the finite theory includes the linear as a limiting case, another condition that must be satisfied by any strain energy function is that on restriction to infinitesimal deformations, the shear and bulk moduli should be positive. Expanding (9) in a Taylor series about the reference configuration and neglecting higher order terms gives

$$\begin{aligned} W = {} & E_{AB}E_{AB}(-\Omega_0'(\tfrac{1}{2}\alpha+\tfrac{3}{2}\beta+\gamma)-\tfrac{1}{2}c_1-\tfrac{3}{2}c_2-c_3) \\ & +E_{AA}E_{BB}(\tfrac{1}{2}\Omega_0'(\beta+\gamma)+\Omega_0''(\alpha+2\beta+\gamma)^2+\tfrac{1}{2}c_2+\tfrac{1}{2}c_3) \end{aligned} \tag{12}$$

where we have employed the summation convention, $\mathbf{E}$ is the usual infinitesimal strain tensor and

$$\Omega_0'' \equiv \left.\frac{\mathrm{d}^2\Omega}{\mathrm{d}(\alpha i_1+\beta i_2+\gamma i_3)^2}\right|_{i_1=i_2=3;i_3=1}. \tag{13}$$

Comparing (12) with the linear form of the strain energy function and imposing the standard linear theory restrictions we obtain

$$\Omega_0'(\alpha+3\beta+2\gamma)+c_1+3c_2+2c_3 < 0, \quad 6\Omega_0''(\alpha+2\beta+\gamma)^2+\Omega_0'(\gamma-\alpha)-c_1+c_3 > 0. \tag{14}$$

We note that setting $c_1 = c_2 = c_3 = 0$ is not consistent with the above set of restrictions and therefore we conclude that at least one of c_1, c_2, c_3 must be non-zero.

In the context of large deformations, a number of constitutive inequalities have been proposed as being necessary to ensure physically realistic behaviour. One set of inequalities is the strong tension-extension (or STE) inequalities. As noted in Truesdell and Noll (1965), the physical motivation for these is that when a cube of isotropic material is lengthened along one principal direction while its faces parallel to that direction are kept fixed, the tensile force must be increased, but to shorten it, the tensile force must be reduced. An elastic material satisfies the STE inequalities if, and only if,

$$\frac{\partial t_i}{\partial \lambda_i} > 0, \quad i = 1,2,3, (\text{no sum}), \tag{15}$$

everywhere, where t_i, λ_i are the principal Cauchy stresses and principal stretches respectively. For the GC material, the STE inequalities reduce to:

$$\frac{\mathrm{d}^2\Omega(x)}{\mathrm{d}x^2} > 0, \tag{16}$$

i.e., the function $\Omega(x)$ must be convex. In what follows we will assume that (16) holds.

4. RADIAL SPHERICAL DEFORMATIONS

Deformations having spherical coordinate representation

$$r = \hat{r}(R), \quad \theta = \Theta, \quad \phi = \Phi, \tag{17}$$

with $\mathrm{d}r/\mathrm{d}R \equiv r' > 0$, describe radial expansion or compaction of hollow spheres. The deformation gradient tensor and the stretch tensor have physical components

$$\mathbf{F} = \mathbf{V} = \mathrm{diag}\left(r', \frac{r}{R}, \frac{r}{R}\right). \tag{18}$$

and the principal invariants are given by

$$i_1 = r' + 2\frac{r}{R}, \quad i_2 = \frac{r^2}{R^2} + 2\frac{r}{R}r', \quad i_3 = \frac{r^2}{R^2}r'. \tag{19}$$

For the GC material, the stress response eqn (7) takes the form

$$\mathbf{T} = \Omega'\{\gamma\mathbf{I} + i_3^{-1}\alpha\mathbf{V} + i_3^{-1}i_1\beta\mathbf{V} - i_3^{-1}\beta\mathbf{V}^2\} + c_3\mathbf{I} + i_3^{-1}c_1\mathbf{V} + i_3^{-1}i_1c_2\mathbf{V} - i_3^{-1}c_2\mathbf{V}^2. \tag{20}$$

Substitution from eqns (18) and (19) in the stress response eqn (20) gives the principal stresses as

$$T_{rr} = \Omega'\left\{\gamma + \alpha\frac{R^2}{r^2} + 2\beta\frac{R}{r}\right\} + c_3 + c_1\frac{R^2}{r^2} + 2c_2\frac{R}{r} \tag{21}$$

and

$$T_{\theta\theta} = T_{\phi\phi} = \Omega'\left\{\gamma + \alpha\frac{R}{rr'} + \beta\left(\frac{R}{r} + \frac{1}{r'}\right)\right\} + c_3 + c_1\frac{R}{rr'} + c_2\left(\frac{R}{r} + \frac{1}{r'}\right). \tag{22}$$

The equations of equilibrium reduce to

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0. \tag{23}$$

Substituting (21), (22) into this equation yields

$$\Omega''\left(\gamma + \alpha\frac{R^2}{r^2} + 2\beta\frac{R}{r}\right)\frac{\mathrm{d}}{\mathrm{d}R}(\alpha i_1 + \beta i_2 + \gamma i_3) = 0. \tag{24}$$

Since, by the STE inequalities, Ω is convex and by setting

$$\gamma + \alpha\frac{R^2}{r^2} + 2\beta\frac{R}{r} = 0, \tag{25}$$

we will not be able to satisfy natural boundary conditions, we see that (24) is equivalent to

$$\alpha i_1 + \beta i_2 + \gamma i_3 = s_1, \tag{26}$$

where s_1 is an arbitrary constant. Substitution of (19) into (26) yields a first order ordinary differential equation in r which may be integrated to yield

$$\gamma r^3 + 3\beta r^2 R + 3\alpha r R^2 = s_1 R^3 + s_2, \tag{27}$$

where s_2 is another constant of integration. (27) may now be solved by radicals to obtain the explicit form of the radial deformation field. However, for our present purposes, (27) is more convenient to use.

5. RADIAL DEFORMATIONS OF A SOLID SPHERE

Attention is now focused on radial deformations of a solid sphere with initial radius R_0 which is subjected to a prescribed deformation of amount Λ on its outer surface:

$$r(R_o) = \Lambda R_o, \Lambda > 1. \tag{28}$$

Thus, setting $R = R_o$ in the deformation field (27) implies

$$s_1 = 3\alpha\Lambda + 3\beta\Lambda^2 + \gamma\Lambda^3 - s_2/R_o^3, \tag{29}$$

giving the relationship between the two parameters s_1 and s_2.

Another condition is necessary to completely determine the field (27). A natural and obvious condition to impose is that the sphere remains intact. This yields $s_2 = 0$ in (27) and

$$r(0) = 0. \tag{30}$$

The resulting solution is the homogeneous expansion of the sphere

$$r(R) = \Lambda R. \tag{31}$$

An alternative mode of radial deformation was proposed by Ball (1982). Instead of pure volumetric expansion (31) keeping the sphere intact, a spherical hole may appear at the centre of the sphere. In mathematical terms this is expressed as

$$r(0) \equiv r_c > 0, \tag{32}$$

where r_c is identified as the radius of the initiated cavity. Evaluation of (27) at $R = 0$ now yields

$$s_2 = \gamma r_c^3, \tag{33}$$

and in what follows we will assume that

$$\gamma \neq 0. \tag{34}$$

Although (33) provides an interpretation for s_2, we must impose a further condition in order for it to be determined. A natural boundary condition for a cavitating solution is that the cavity be traction-free

$$T_{rr}|_{R=0} = 0. \tag{35}$$

Using (21), (35) is seen to reduce for the GC material to

$$\frac{d\Omega}{dx}(s_1) = -\frac{c_3}{\gamma}. \tag{36}$$

When it is assumed that a solution to (36) for s_1 exists, it is unique due to the convexity of $\Omega(x)$. Once s_1 is determined from (36), (29) determines s_2 for a prescribed Λ.

Furthermore, combining (29) and (33), we can obtain the following relation between the cavity radius r_c and the applied stretch Λ:

$$\left(\frac{r_c}{R_o}\right)^3 = 3\varepsilon_1\Lambda + 3\varepsilon_2\Lambda^2 + \Lambda^3 - \frac{s_1}{\gamma}, \tag{37}$$

where s_1 is the solution to (36) and

$$\varepsilon_1 \equiv \frac{\alpha}{\gamma}, \quad \varepsilon_2 \equiv \frac{\beta}{\gamma}. \tag{38}$$

We will call those values for Λ for which $r_c = 0$, critical values. Cavitation occurs for those values of Λ for which $r_c > 0$. If for some values of Λ, $r_c < 0$, then, of course, cavitation is not possible.

We first note that the results of Horgan (1992) are recovered in the above analysis on setting $\alpha = \beta = c_3 = 0$, $\gamma = 1$ and that the elegance and compactness of his results are maintained in our generalisation. However, we see that the qualitative features of the physical parameter of interest in cavitation, r_c, can differ significantly from those observed by Horgan who found that beyond a critical value of stretch, the void radius increases monotonically with applied stretch.

We now wish to emphasise that (37) is a *closed form* solution to the problem of determining the relationship between the void radius and the stretch applied at the outer curved surface of the sphere for the general compressible material with the strain energy function given by (9). However we see that the qualitative features of this relationship depend on the specific forms of the arbitrary function and the arbitrary constants defining the strain energy function. This relationship will be considered in Section 7.

6. REMARKS ON ENERGY FUNCTIONALS

In order to determine which of the homogeneous or the cavitating modes of deformation is preferred for the GC material, associated energy functionals are examined. Since the displacement on the outer surface is prescribed and the cavity surface (if it exists) is traction-free, the associated potential energy, E, is given by

$$E = \int_V W\,\mathrm{d}V = 4\pi\int_0^{R_0} WR^2\,\mathrm{d}R. \tag{39}$$

Using the equation of equilibrium for radial deformations, (39) can be rearranged as

$$E = \frac{4\pi R_o^3}{3}\left\{W - (r'(R_o) - \Lambda)\frac{\partial W}{\partial \lambda_1}\right\}, \tag{40}$$

where both W and $\partial W/\partial\lambda_1$ are evaluated at $R = R_o$. This is the same form of the energy functional used by Horgan (1992).

Since for homogeneous solution $r(R) = \Lambda R$, it is seen that the potential energy for the homogeneous mode, E_H, is given by

$$E_H/(4\pi R_o^3/3) = \Omega(3\alpha\Lambda + 3\beta\Lambda^2 + \gamma\Lambda^3) + 3c_1\Lambda + 3c_2\Lambda^2 + c_3\Lambda^3. \tag{41}$$

A straightforward manipulation involving (40) and (26) yields the corresponding functional for the cavitating solution, E_c, as

$$E_c/(4\pi R_o^3/3) = \Omega(s_1) + 3c_1\Lambda + 3c_2\Lambda^2 + c_3\Lambda^3 - \frac{c_3}{\gamma}(3\alpha\Lambda + 3\beta\Lambda^2 + \gamma\Lambda^3 - s_1), \tag{42}$$

where (36) was used to evaluate $\Omega'(s_1)$.

Thus the difference between the two types of solution, normalised by the reference volume of the sphere, becomes

$$\tilde{E}(\xi) = \frac{E_c - E_H}{4\pi R_o^3/3} = \Omega(s_1) - \Omega(\xi) + \frac{c_3}{\gamma}(s_1 - \xi), \tag{43}$$

where an auxiliary parameter was introduced defined by

$$\xi = 3\alpha\Lambda + 3\beta\Lambda^2 + \gamma\Lambda^3 = \gamma(r_c/R_o)^3 + s_1. \tag{44}$$

Thus a cavitating solution implies $\xi > s_1$. It can be readily observed that the difference in (43) vanishes as $\xi \to s_1$, i.e.,

$$\tilde{E}(\xi)|_{\xi \to s_1} = 0. \tag{45}$$

We proceed to take the derivative of the above difference with respect to ξ as follows

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{E}(\xi) = -\frac{\mathrm{d}\Omega}{\mathrm{d}\xi}(\xi) - \frac{c_3}{\gamma}. \tag{46}$$

Noting (36), the above derivative is also seen to vanish as $\xi \to s_1$:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{E}(\xi)|_{\xi \to s_1} = -\frac{\mathrm{d}\Omega}{\mathrm{d}\xi}(s_1) - \frac{c_3}{\gamma} = 0. \tag{47}$$

Therefore, as it has been assumed that the function $\Omega(x)$ is convex, it is clear that the energy difference is negative for $\xi > s_1$ while it is positive for $\xi < s_1$. Thus in terms of potential energy, the cavitating mode is preferred to the homogeneous deformation when the cavitating solution exists with a cavity of positive radius.

7. SOME QUALITATIVE FEATURES

In Section 5 the relation between the applied stretch Λ and the radius of an initiated cavity, r_c, is given in (37) for the GC material. Using this, we can trace the growth behaviour of the cavity as the stretch is increased. In order to investigate the influence of the employed material parameters on this behaviour, it is convenient to rewrite (37) as

$$\left(\frac{r_c}{R_o}\right)^3 = f(\Lambda) - \frac{s_1}{\gamma} = \Lambda^3 + 3\varepsilon_2\Lambda^2 + 3\varepsilon_1\Lambda - \frac{s_1}{\gamma}. \tag{48}$$

As pointed out, e.g., by Horgan and Abeyaratne (1986), cavitation is an inherently nonlinear phenomenon and cannot be modelled using linearised solid mechanics theories. Since linear elasticity corresponds in (48) to the limiting case where $\Lambda \to 1$, we will assume that a (unique) solution to (36) exists such that

$$f(1) - \frac{s_1}{\gamma} = 3\varepsilon_1 + 3\varepsilon_2 + 1 - \frac{s_1}{\gamma} < 0 \tag{49}$$

is satisfied. This yields that $r_c < 0$ when $\Lambda = 1$. Also it is immediate from (48) that as the stretch Λ is increased without bound, $r_c \sim \Lambda R_o$. For the intermediate range of stretch, the behaviour of the cavity radius depends on our choice of material parameters ε_1 and ε_2 in (48). Below, some qualitative features of this behaviour will be discussed.

To this end, first it is noted that $f(\Lambda)$ in (48) has at most two extrema which satisfy

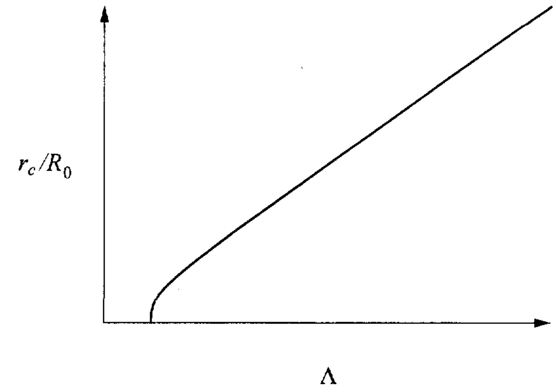

Fig. 1. The relation between the applied stretch and the cavity radius (mode I).

$$\frac{1}{3}\frac{df}{d\Lambda} = \Lambda^2 + 2\varepsilon_2\Lambda + \varepsilon_1 = 0. \tag{50}$$

Different types of behaviour can thus be classified based on the discriminant of the above quadratic equation. Each case will be considered separately.

(i) $\varepsilon_2^2 < \varepsilon_1$

In this case (50) does not possess real roots and the cavity radius therefore grows monotonically as the stretch increases (Fig. 1). This monotonic dependence of r_c upon Λ will be termed mode I in what follows. We note that mode I behaviour was observed for the Varga material by Horgan (1992) and for a special Blatz–Ko material by Horgan and Abeyaratne (1986). For incompressible materials, mode I is the only mode possible due to the internal constraint. It appears to be the only mode of cavitation observed so far in the literature.

(ii) $\varepsilon_2^2 = \varepsilon_1$

In this case (50) is satisfied by only one value of Λ, i.e., $\Lambda = -\varepsilon_2$. Consequently, for this case, mode I is also the qualitative behaviour of the cavity radius.

(iii) $\varepsilon_2^2 > \varepsilon_1$

In this case further analysis is required in order to predict the behaviour of the cavity radius. We first note that (50) has two solutions

$$\Lambda_1 = -\varepsilon_2 - \sqrt{\varepsilon_2^2 - \varepsilon_1}, \quad \Lambda_2 = -\varepsilon_2 + \sqrt{\varepsilon_2^2 - \varepsilon_1}. \tag{51}$$

Λ_1 yields a maximum value, Λ_2 a minimum and $\Lambda_1 < \Lambda_2$.

It is easy to show that mode I behaviour is again observed when $\Lambda_1 \leqslant 1$ and therefore a different mode of cavitation can occur only when $\Lambda_1 > 1$. This condition is equivalent to

$$-\tfrac{1}{2}(1+\varepsilon_1) < \varepsilon_2 < -1. \tag{52}$$

This region in the $\varepsilon_1\varepsilon_2$ plane is shown in Fig. 2.

We will now confine our attention to the shaded region in Fig. 2. There are three cases:

1. $s_1/\gamma > f(\Lambda_1)$.

From (48), we see that $r_c|_{\Lambda=\Lambda_1} < 0$. Noting the asymptotic behaviour of r_c, we conclude that in this case mode I is again observed.

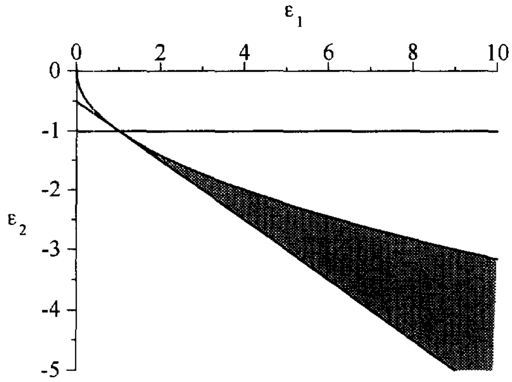

Fig. 2. The region where modes II and III are possible.

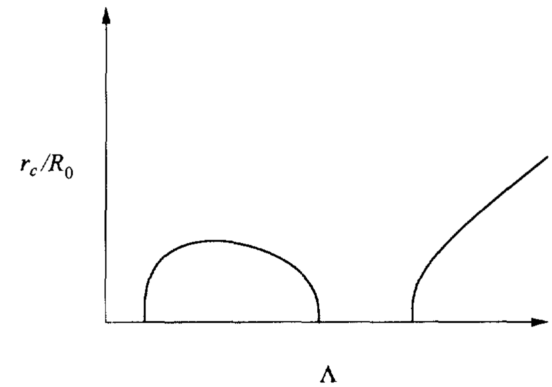

Fig. 3. The relation between the applied stretch and the cavity radius (mode II).

2. $f(\Lambda_2) < s_1/\gamma < f(\Lambda_1)$

For this case, $r_c|_{\Lambda=\Lambda_1} > 0$ and $r_c|_{\Lambda=\Lambda_2} < 0$. In this case therefore as the stretch is increased from 1, r_c increases from a negative value to reach a positive maximum at $\Lambda = \Lambda_1$, then decreases to a negative minimum at $\Lambda = \Lambda_2$ and finally increases unboundedly. A physical interpretation of this is that as the stretch increases, a cavity first appears, then collapses and finally reappears again. This type of cavitation we will call mode II and a typical plot of cavity radius vs stretch for mode II is given in Fig. 3.

3. $s_1/\gamma < f(\Lambda_2)$

For this case, $r_c|_{\Lambda=\Lambda_1} > 0$ and $r_c|_{\Lambda=\Lambda_2} > 0$. In this case therefore as the stretch is increased from 1, r_c increases from a negative value to reach a positive maximum at $\Lambda = \Lambda_1$, then decreases to a positive minimum at $\Lambda = \Lambda_2$ and finally increases unboundedly. A physical interpretation of this is that as the stretch increases, a cavity first appears, then after a period of strain softening behaviour, the cavity grows unboundedly. This type of cavitation we will call mode III and a typical plot of cavity radius vs stretch for mode II is given in Fig. 4.

Mode I cavitation, implying monotonic growth of the cavity with increasing applied stretch, has been frequently demonstrated in previous work, e.g., Horgan (1992), Horgan and Abeyaratne (1986), Ertan (1988) and Biwa *et al.* (1994), to name but a few. The other two modes, II and III, appear new and, we believe, warrant due attention. For the GC material under study, the mode of cavitation is determined by the choice of material parameters ε_1 and ε_2. We wish to emphasise that the cavitating mode is always preferred to the homogeneous deformation once a cavity appears regardless of mode of cavitation, using the potential energy criterion.

The most general class of strain energy functions for which (27) describes the corresponding radial deformation field was obtained by Murphy (1997). This general class

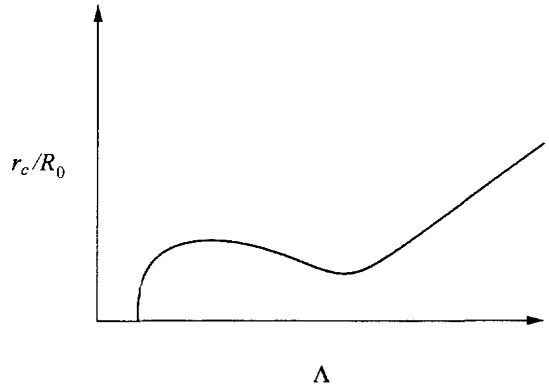

Fig. 4. The relation between the applied stretch and the cavity radius (mode III).

includes the strain energy function assumed at the outset, but most generally it is described completely by the following form:

$$W = \Omega(x) + c_1 i_1 + c_2 i_2 + c_3 i_3 + \Psi(y)(i_1 - 3y) + 3\int \Psi(y)\, dy, \tag{53}$$

where

$$x = \alpha i_1 + \beta i_2 + \gamma i_3, \quad y = \frac{i_1 \pm \sqrt{i_1^2 - 3i_2}}{3} \tag{54}$$

and $\Psi(y)$ is another arbitrary function. In particular, it is easy to show that the strain energy function with $\alpha = \beta = 0$ is the most general strain energy function for which the cavitation solution of Horgan (1992) is valid. We note that all of the above analysis can be extended to this most general class of materials (53) although some of the conciseness of the above results would be lost without any greater insight into the problem at hand.

Finally we remark that cavitation for the GC material can be interpreted as the growth of a negligibly small pre-existing micro-void but we will not undertake such analysis here.

8. RADIAL CYLINDRICAL DEFORMATIONS

Deformations having cylindrical coordinate representation

$$r = \hat{r}(R), \quad \theta = \Theta, \quad z = \lambda Z, \tag{55}$$

where $dr/dR \equiv r' > 0$ and $\lambda > 0$, describe radial deformations of cylindrical bodies accompanied by an axial stretch. The deformation gradient tensor and the stretch tensor have physical components

$$\mathbf{F} = \mathbf{V} = \operatorname{diag}\left(r', \frac{r}{R}, \lambda\right), \tag{56}$$

and the principal invariants are

$$i_1 = r' + \frac{r}{R} + \lambda, \quad i_2 = \left(\frac{r}{R} + \lambda\right) r' + \lambda \frac{r}{R}, \quad i_3 = \lambda \frac{r}{R} r'. \tag{57}$$

We will confine our attention in what follows to the GC material. A similar analysis can be undertaken for the more general material (53) but will not be done here. Substitution from eqns (56) and (57) into the stress response eqn (20) for the GC material yields

$$T_{rr} = \left(\gamma+\alpha\frac{R}{\lambda r}+\beta\left(\frac{1}{\lambda}+\frac{R}{r}\right)\right)\Omega'+c_3+c_1\frac{R}{\lambda r}+c_2\left(\frac{1}{\lambda}+\frac{R}{r}\right), \tag{58}$$

$$T_{\theta\theta} = \left(\gamma+\alpha\frac{1}{\lambda r'}+\beta\left(\frac{1}{\lambda}+\frac{1}{r'}\right)\right)\Omega'+c_3+c_1\frac{1}{\lambda r'}+c_2\left(\frac{1}{\lambda}+\frac{1}{r'}\right), \tag{59}$$

and

$$T_{zz} = \left(\gamma+\alpha\frac{R}{rr'}+\beta\left(\frac{1}{r'}+\frac{R}{r}\right)\right)\Omega'+c_3+c_1\frac{R}{rr'}+c_2\left(\frac{1}{r'}+\frac{R}{r}\right). \tag{60}$$

The axial and azimuthal equations of equilibrium are satisfied identically and the radial equation

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}+\frac{1}{r}(T_{rr}-T_{\theta\theta}) = 0 \tag{61}$$

reduces to the form

$$\frac{1}{r'}\left(\gamma+\alpha\frac{R}{\lambda r}+\beta\left(\frac{1}{\lambda}+\frac{R}{r}\right)\right)\Omega''\frac{\mathrm{d}}{\mathrm{d}R}(\alpha i_1+\beta i_2+\gamma i_3) = 0. \tag{62}$$

Arguing here as in Section 4 shows that (62) is equivalent to

$$\alpha i_1+\beta i_2+\gamma i_3 = s_1, \tag{63}$$

where s_1 is a constant of integration. Substituting from (57) into (63) yields a first order equation in r which is easily integrated to obtain

$$r^2(\beta+\gamma\lambda)+2rR(\alpha+\beta\lambda) = (s_1-\alpha\lambda)R^2+s_2, \tag{64}$$

where s_2 is a constant of integration. This form of the radial deformation field is especially convenient for the purposes of our study. Eqn (64) can, of course, be easily solved to obtain the radial deformation field explicitly.

9. DEFORMATIONS OF SOLID CYLINDERS

Consider a solid cylinder of undeformed radius R_o subjected to an axial stretch of amount λ and also subjected to a radial stretch of amount μ at the curved surface. Thus we will require that

$$r(R_o) = \mu R_o, \tag{65}$$

where $\mu > 1$.

As in the spherical case, requiring that the cylinder remains solid yields

$$r = \mu R, \tag{66}$$

the homogeneous deformation field.

Suppose now that a void appears at the centre of the cylinder. Thus

$$r(0) \equiv r_c > 0, \tag{67}$$

and imposing the condition that the cavity be stress free yields

$$T_{rr}|_{R=0} = 0. \tag{68}$$

Applying (64) at the outer boundary and using (65) gives

$$s_2 = R_o^2\{\mu^2(\beta+\gamma\lambda)+2\mu(\alpha+\beta\lambda)+\alpha\lambda-s_1\}. \tag{69}$$

Applying (64) at $R = 0$, we obtain

$$r_c^2(\beta+\gamma\lambda) = s_2. \tag{70}$$

Assuming that $\beta+\gamma\lambda \neq 0$, we obtain from (69), (70)

$$\left(\frac{r_c}{R_o}\right)^2 = \mu^2+2\mu m_c+\frac{\alpha\lambda-s_1}{\beta+\gamma\lambda}, \tag{71}$$

where

$$m_c \equiv \frac{\alpha+\beta\lambda}{\beta+\gamma\lambda}. \tag{72}$$

In the above, s_1 is determined from the stress-free boundary condition (68). This reduces to the equation

$$\Lambda'(s_1) = -\frac{c_2+\lambda c_3}{\beta+\gamma\lambda}. \tag{73}$$

We again note that the results of Horgan (1992) can be recovered in this case. Setting $\alpha = \beta = 0$, $\gamma = \lambda = 1$ and $c_2 = c_3 = 0$ in the above analysis gives the same results.

We now focus on the qualitative behaviour of r_c. Consideration of the asymtotics in (71) and the behaviour of r_c as $\mu \to 1$, shows that for cylindrical cavitation only one mode is possible for the GC family of materials: that after a critical value of the stretch, the cavity radius increases monotonically with applied radial stretch. Thus mode I behaviour is the only possible means of cavitation for cylinders composed of the GC material. This mode of cylindrical cavitation has previously been observed for the Varga material by Horgan (1992) and for a special Blatz-Ko material by Horgan and Abeyaratne (1986).

Proceeding as for the spherical case, it is easy to show that cavitation (where possible) is preferred over the homogeneous deformation for the GC material.

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